Wednesday, September 2, 2015

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Problem 1

Problem. Find the indefinite integral $\int \frac{5}{x}\,dx$.

Solution.

$$\int \frac{5}{x}\,dx = 5\ln|x| + C.$$

Problem 3

Problem. Find the indefinite integral $\int \frac{1}{x+1}\,dx$.

Solution. Let $u = x + 1$ and $du = dx$. Then we get

$$\begin{aligned}\int \frac{1}{x+1}\,dx &= \int \frac{1}{u}\,du \\ &= \ln|u| + C \\ &= \ln|x+1| + C.\end{aligned}$$

Problem 7

Problem. Find the indefinite integral $\int \frac{x}{x^2-3}\,dx$.

Solution. Let $u = x^2 - 3$ and $du = 2x\,dx$. Then we get

$$\begin{aligned}\int \frac{x}{x^2-3}\,dx &= \frac{1}{2}\int \frac{2x}{x^2-3}\,dx \\ &= \frac{1}{2}\int \frac{1}{u}\,du \\ &= \frac{1}{2}\ln|u| + C \\ &= \frac{1}{2}\ln|x^2-3| + C.\end{aligned}$$

Problem 12

Problem. Find the indefinite integral $\int \frac{x^3-8x}{x^2}\,dx$.

Solution. First, divide x^2 into each term of the numerator. Then integrate termwise.

$$\int \frac{x^3 - 8x}{x^2}\,dx = \int \left(x - \frac{8}{x}\right)\,dx$$
$$= \frac{1}{2}x^2 - 8\ln|x| + C.$$

Problem 13

Problem. Find the indefinite integral $\displaystyle\int \frac{x^2+2x+3}{x^3+3x^2+9x}\,dx$.

Solution. Let $u = x^3 + 3x^2 + 9x$ and $du = (3x^2 + 6x + 9)\,dx$, which happens to be 3 times the numerator. Then we get

$$\int \frac{x^2+2x+3}{x^3+3x^2+9x}\,dx = \frac{1}{3}\int \frac{3x^2+6x+9}{x^3+3x^2+9x}\,dx$$
$$= \frac{1}{3}\int \frac{1}{u}\,du$$
$$= \frac{1}{3}\ln|u| + C$$
$$= \frac{1}{3}\ln|x^3+3x^2+9x| + C.$$

Problem 17

Problem. Find the indefinite integral $\displaystyle\int \frac{x^3-3x^2+5}{x-3}\,dx$.

Solution. Begin by using long division to divide $x-3$ into x^3-3x^2+5. We get x^2 with a remainder of 5, so

$$\frac{x^3-3x^2+5}{x-3} = x^2 + \frac{5}{x-3}.$$

Now we can integrate.

$$\int \frac{x^3-3x^2+5}{x-3}\,dx = \int \left(x^2 + \frac{5}{x-3}\right)\,dx$$
$$= \frac{1}{3}x^3 + 5\ln|x-3| + C.$$

Problem 27

Problem. Find the indefinite integral $\displaystyle\int \frac{1}{1+\sqrt{2x}}\,dx$ by u-substitution.

Solution. Let $u = 1+\sqrt{2x}$ and $du = \frac{1}{\sqrt{2x}}\,dx$. Note that $du = \frac{1}{u}\,dx$, so we may rewrite that as $dx = u\,du$ and use that for the substitution.

$$\begin{aligned}
\int \frac{1}{1+\sqrt{2x}}\,dx &= \int \left(\frac{1}{1+u}\right)(u\,du) \\
&= \int \frac{u}{1+u}\,du \\
&= \int \left(1 - \frac{1}{1+u}\right)\,du \\
&= u - \ln|1+u| + C \\
&= \sqrt{2x} - \ln|1+\sqrt{2x}| + C.
\end{aligned}$$

Problem 29

Problem. Find the indefinite integral $\displaystyle\int \frac{\sqrt{x}}{\sqrt{x}-3}\,dx$ by u-substitution.

Solution. Let $u = \sqrt{x}$ and $du = \frac{1}{2\sqrt{x}}\,dx$. Note that $du = \frac{1}{2u}\,dx$, so $dx = 2u\,du$. Now substitute and integrate.

$$\begin{aligned}
\int \frac{\sqrt{x}}{\sqrt{x}-3}\,dx &= \int \left(\frac{u}{u-3}\right)\cdot 2u\,du \\
&= \int \left(\frac{2u^2}{u-3}\right)\,du \\
&= \int \left(2u + 6 + \frac{18}{u-3}\right)\,du \\
&= u^2 + 6u + 18\ln|u-3| + C \\
&= x + 6\sqrt{x} + 18\ln|\sqrt{x}-3| + C.
\end{aligned}$$

Or, you could let $u = \sqrt{x}-3$ and $du = \frac{1}{2\sqrt{x}}\,dx$. Then $\sqrt{x} = u+3$ and $du = \frac{1}{2(u+3)}\,dx$, so $dx = 2(u+3)\,du$. Now substitute and integrate.

$$\begin{aligned}
\int \frac{\sqrt{x}}{\sqrt{x}-3}\,dx &= \int \left(\frac{u+3}{u}\right) \cdot 2(u+3)\,du \\
&= 2\int \left(\frac{(u+3)^2}{u}\right)\,du) \\
&= 2\int \left(\frac{u^2+6u+9}{u}\right)\,du) \\
&= 2\int \left(u+6+\frac{9}{u}\right)\,du) \\
&= u^2 + 12u + 18\ln|u| + C \\
&= (\sqrt{x}-3)^2 + 12(\sqrt{x}-3) + 18\ln|\sqrt{x}-3| + C.
\end{aligned}$$

Problem 32

Problem. Find the indefinite integral $\int \tan 5\theta\,d\theta$.

Solution. Let $u = 5\theta$ and $du = 5\,d\theta$. Then

$$\begin{aligned}
\int \tan 5\theta\,d\theta &= \frac{1}{5}\int 5\tan 5\theta\,d\theta \\
&= \frac{1}{5}\int \tan u\,du \\
&= \frac{1}{5}\ln|\sec u| + C \\
&= \frac{1}{5}\ln|\sec 5\theta| + C.
\end{aligned}$$

Problem 34

Problem. Find the indefinite integral $\int \sec\frac{x}{2}\,dx$.

Solution. Let $u = \frac{x}{2}$ and $du = \frac{1}{2}\,dx$. Then

$$\begin{aligned}
\int \sec\frac{x}{2}\,dx &= 2\int \frac{1}{2}\sec\frac{x}{2}\,dx \\
&= 2\int \sec u\,du \\
&= 2\ln|\sec u + \tan u| + C \\
&= 2\ln\left|\sec\frac{x}{2} + \tan\frac{x}{2}\right| + C.
\end{aligned}$$

Problem 39

Problem. Find the indefinite integral $\int \frac{\sec x \tan x}{\sec x - 1}\, dx$.

Solution. Let $u = \sec x - 1$ and $du = \sec x \tan x\, dx$. Then

$$\begin{aligned}\int \frac{\sec x \tan x}{\sec x - 1}\, dx &= \int \frac{du}{u} \\ &= \ln|u| + C \\ &= \ln|\sec x - 1| + C.\end{aligned}$$

Problem 70

Problem. Find the area of the region under $y = \frac{\sin x}{1+\cos x}$ from $x = \frac{\pi}{4}$ to $x = \frac{3\pi}{4}$.

Solution. Let $u = 1 + \cos x$ and $du = -\sin x\, dx$. Then $u\left(\frac{\pi}{4}\right) = 1 + \frac{1}{\sqrt{2}}$ and $u\left(\frac{3\pi}{4}\right) = 1 - \frac{1}{\sqrt{2}}$.

$$\begin{aligned}\int_{\pi/4}^{3\pi/4} \frac{\sin x}{1+\cos x}\, dx &= -\int_{\pi/4}^{3\pi/4} \frac{-\sin x}{1+\cos x}\, dx \\ &= -\int_{1+\frac{1}{\sqrt{2}}}^{1-\frac{1}{\sqrt{2}}} \frac{du}{u} \\ &= -\left[\ln|u|\right]_{1+\frac{1}{\sqrt{2}}}^{1-\frac{1}{\sqrt{2}}} \\ &= -\ln\left|1 - \frac{1}{\sqrt{2}}\right| + \ln\left|1 + \frac{1}{\sqrt{2}}\right| \\ &= \ln\left|\frac{1+\frac{1}{\sqrt{2}}}{1-\frac{1}{\sqrt{2}}}\right| \\ &= \ln\left|\frac{\sqrt{2}+1}{\sqrt{2}-1}\right|.\end{aligned}$$

Problem 96

Problem. Find the average value of $f(x) = \sec \frac{\pi x}{6}$ over the interval $[0, 2]$.

Solution. The formula for the average value of a function $f(x)$ over an interval $[a, b]$ is

$$\frac{1}{b-a}\int_a^b f(x)\, dx.$$

So we need to find the value of

$$\frac{1}{2}\int_0^2 \sec\frac{\pi x}{6}\,dx.$$

Let $u = \frac{\pi x}{6}$ and $du = \frac{\pi}{6}\,dx$. Then $u()0) = 0$ and $u(2) = \frac{\pi}{3}$.

$$\begin{aligned}
\frac{1}{2}\int_0^2 \sec\frac{\pi x}{6}\,dx &= \frac{6}{\pi}\cdot\frac{1}{2}\int_0^2 \frac{\pi}{6}\sec\frac{\pi x}{6}\,dx \\
&= \frac{3}{\pi}\int_0^{\pi/3} \sec u\,du \\
&= \frac{3}{\pi}\left[\ln|\sec u + \tan u|\right]_0^{\pi/3} \\
&= \frac{3}{\pi}\left(\ln\left|\sec\frac{\pi}{3} + \tan\frac{\pi}{3}\right| - \ln|\sec 0 + \tan 0|\right) \\
&= \frac{3}{\pi}\left(\ln\left|2+\sqrt{3}\right| - \ln 1\right) \\
&= \frac{3}{\pi}\ln\left(2+\sqrt{3}\right).
\end{aligned}$$

Problem 97

Problem. A population of bacteria P is changing at a rate of

$$\frac{dP}{dt} = \frac{3000}{1+0.25t},$$

where t is the time in days. The initial population (when $t = 0$) is 1000. Write an equation that gives the population at any time t. Then find the population when $t = 3$.

Solution. The population $P(t)$ is given by

$$P(t) = \int \frac{3000}{1+0.25t}\,dt.$$

Let $u = 1 + 0.25t$ and $du = 0.25\,dt$. Then

$$\begin{aligned}
P(t) &= \int \frac{3000}{1+0.25t}\,dt \\
&= 12000\int \frac{0.25}{1+0.25t}\,dt \\
&= 12000\int \frac{du}{u} \\
&= 12000\ln|u| + C \\
&= 12000\ln|1+0.25t| + C.
\end{aligned}$$

To find the value of C, let $t = 0$.

$$\begin{aligned} P(0) &= 12000 \ln 1 + C \\ &= C. \end{aligned}$$

Therefore, $C = 1000$ and the population is

$$P(t) = 12000 \ln (1 + 0.25t) + 1000.$$

When $t = 3$, the population is

$$\begin{aligned} P(3) &= 12000 \ln 1.75 + 1000 \\ &= 12000(0.5596) + 1000 \\ &= 7715. \end{aligned}$$